Minimum mass-radius ratio for charged gravitational objects

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Abstract

We rigorously prove that for compact charged general relativistic objects there is a lower bound for the mass-radius ratio. This result follows from the same Buchdahl type inequality for charged objects, which has been extensively used for the proof of the existence of an upper bound for the mass-radius ratio. The effect of the vacuum energy (a cosmological constant) on the minimum mass is also taken into account. Several bounds on the total charge, mass and the vacuum energy for compact charged objects are obtained from the study of the Ricci scalar invariants. The total energy (including the gravitational one) and the stability of the objects with minimum mass-radius ratio is also considered, leading to a representation of the mass and radius of the charged objects with minimum mass-radius ratio in terms of the charge and vacuum energy only.

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I. INTRODUCTION

The bag models of hadrons [1], proposed in the 1970's, have had a remarkable phenomenological success (see [2] and [3] for reviews and recent developments). In these models, hadrons consist of free (or only weekly interacting) quarks, which are confined to a finite region of space, called the bag. The confinement is not a dynamical one, but it is put in by hand, imposing some appropriate boundary conditions. The bag is stabilised by a term of the form $g_{\mu\nu}B$, which is added to the energy momentum tensor $T_{\mu\nu}$ inside the bag, which thus takes the form $T_{\mu\nu} = T_{\mu\nu}^{(\text{fields})} + g_{\mu\nu}B$. By recalling the energy-momentum tensor of a perfect fluid in its rest-frame, $T^{\mu}_{\nu} = \text{diag}\,(\epsilon, -p, -p, -p)$, where ϵ is the energy density and p is the thermodynamic pressure, it immediately follows that the bag constant B is immediately interpreted as positive contribution to the energy density ϵ and a negative contribution to the pressure p inside the bag. Equivalently, we may attribute a term $-g_{\mu\nu}B$ to the region outside the bag. This leads to a picture of a non-trivial vacuum with a negative energy density $\epsilon_{vac} = -B$ and a positive pressure $p_{vac} = +B$. The stability of the hadron then results from balancing this positive vacuum pressure with the pressure caused by the quarks inside the bag [3].

Therefore, quark bag models in the theories of strong interactions assume that the breaking of physical vacuum takes place inside hadrons. As a result the vacuum energy densities inside and outside a hadron become essentially different and the vacuum pressure B on a bag wall equilibrates the pressure of quarks thus stabilising the system. The MIT bag model says nothing about the origin of the non-trivial vacuum, but treats B as a free parameter. Assuming a static spherical bag of radius R, the mass of the hadron is given by the sum $E_{BM} = 4\pi BR^3/3 - z_0/R + \sum_q x_q/R + ...$, where the first term corresponds to the volume energy, required to replace the non-trivial vacuum by the trivial one inside the bag, the second term parameterise the finite part of the zero-point energy of the bag and the third term is the sum of the rest and kinetic energy of the quarks [3].

The finite electron self-energy is a puzzling problem in both quantum theory and classical theory. Quantum electrodynamics, with its remarkable predictive power, fails to explain the origin of the finite electron mass, and none of the proposed regularisation schemes have succeeded in predicting the observed mass. On the other hand, a point charge is incompatible with classical electrodynamics, because it has the self-energy and stability problems. An electron of finite radius was proposed by Abraham and Lorentz, with the particle radius equal to $R = Q^2/M$, where Q and M are the charge and the mass of the particle, respectively. This relation has been obtained by assuming that the electromagnetic potential energy of the particle Q^2/R is equal to its mass M, according to the mass-energy equivalence law. However, an extended charge distribution interacting with itself cannot be stable and non-electromagnetic forces are needed to prevent the electron from exploding. Such cohesive non-electromagnetic forces were suggested by Poincaré, and are called Poincaré stresses [4].

On the other hand, the Einstein-Maxwell field equations of general relativity can be used to construct a Lorentz model of an electron as an extended body consisting of pure charge and no matter and electromagnetic mass models for static spherically symmetric charged fluid distributions have been extensively studied [5]. The Poincaré stresses are explained as due to vacuum polarisation, the vacuum energy density ρ_V and the vacuum pressure p_V satisfying an equation of state of the form $\rho_V + p_V = 0$, where in general the vacuum energy density $\rho_V > 0$ and the pressure $p_V < 0$. This type of equation of state implies that the matter distribution under consideration is in tension, in a state known as "false vacuum" or "degenerate vacuum". The gravitational blue-shift of light is explained as due to repulsive gravitation produced by the negative gravitational mass of the polarised vacuum. In the context of general relativity, the electron, modelled as a spherically symmetric charged distribution of matter, must contain some negative rest mass if its radius is not larger than 10^{-16} cm. In some extended electron models, the negative energy density distributions result from the requirement that the total mass of these models remains constant in the limit of a point particle.

The mass-radius-charge relation for elementary particles,compact astrophysical objects or black holes plays an important role in many physical processes. The pressure and the density of the matter inside the stars are large, and the gravitational field is intense. This indicates that electric charge and a strong electric field may also be present. The effect of electric charge in compact stars assuming that the charge distribution is proportional to the mass density was studied in [6]. In order to see any appreciable effect on the phenomenology of the compact stars, the electric fields have to be huge (10^{21} V/m), which implies that the total charge is $Q \approx 10^{20}$ Coulomb. The star can then collapse to form a charged black hole. Charged stars have the potential of becoming charged black holes or even naked singularities. A set of numerical solutions of the Tolman-Oppenheimer-Volkov equations that represents spherical charged compact stars in hydrostatic equilibrium were obtained in [7]. Charged boson stars in scalar-tensor gravitational theories have been studied in [8]. In these models there is a maximum charge to mass ratio for the bosons above which the weak field solutions are not stable. This charge limit can be greater than the general relativistic limit for a wide class of scalar-tensor theories. The black hole formation in the head-on collision of ultra-relativistic charges was studied in [9]. The formation of the apparent horizon was analysed and a condition was obtained, indicating that a critical value of the electric charge is necessary for black hole formation to take place. By evaluating this condition for characteristic values at the LHC, it was found that the presence of the charge decreases the black hole production rate in accelerators. Mass-charge limits are important for the study of the quasi-local energy measured by observers who are moving around in the space-time. The quasi-local formalism for gravitational energy was extended in [10] to include electromagnetic and dilaton fields and to also allow for spatial boundaries that are not orthogonal to the foliation of the space-time. The distribution of energy around Reissner-Nordström and naked black holes was investigated as measured by both static and infalling observers. The study of naked black holes reveals an alternate characterisation of this class of space-times in terms of the quasi-local energies.

The observations of high redshift supernovae [11] and the Boomerang/Maxima data [12], showing that the location of the first acoustic peak in the power spectrum of the microwave background radiation is consistent with the inflationary prediction $\Omega = 1$, have provided compelling evidence for a net equation of state of the cosmic fluid lying in the range $-1 \leq w = p/\rho < -1/3$. To explain these observations, two dark components are invoked: the pressure-less cold dark matter (CDM) and the dark energy (DE) with negative pressure. CDM contributes $\Omega_m \sim 0.25$, and is mainly motivated by the theoretical interpretation of the galactic rotation curves and large scale structure formation. DE provides $\Omega_{DE} \sim 0.7$ and is responsible for the acceleration of the distant type Ia supernovae. The best candidate for the dark energy is the cosmological constant Λ, which is usually interpreted physically as a vacuum energy. Its size is of the order $\Lambda \approx 3 \times 10^{-56}$ cm^{-2} [13].

However, the WMAP data also allow the possibility that the Universe may be slightly above/below the ΛCDM model, in the so called phantom region (see [14] and references therein). In the phantom scenario the acceleration of the Universe is explained by the presence of some phantom matter, with negative energy density. The similarity of phantom matter with quantum CFT indicates that the phantom scalar may be the effective description for some quantum field theory [14]. For phantom/tachyonic matter the standard energy conditions of general relativity, the null energy condition (NEC) $\rho + p \geq 0$, the weak energy condition (WEC) $\rho \geq 0$ and $\rho + p \geq 0$, the strong energy condition (SEC) $\rho + 3p \geq 0$ and $\rho + p \geq 0$ and dominant energy condition (DEC) $\rho \geq 0$ and $\rho \pm p \geq 0$ are violated [15]. Such a model naturally admits two de Sitter phases where the early universe inflation is produced by quantum effects and the late time accelerating universe is caused by phantom/tachyon. The typical final state of a dark energy universe where a dominant energy condition is violated is a finite-time, sudden future singularity (a big rip). For a number of dark energy universes (including scalar phantom and effective phantom theories as well as specific quintessence models) the quantum effects play the dominant role near a big rip, driving the universe out of a future singularity [16]. Black hole mass loss due to phantom accretion is very different from the standard general relativistic case: masses do not vanish to zero due to the transient character of the phantom evolution stage [16].

By using the static spherically symmetric gravitational field equations Buchdahl [17] has obtained an absolute constraint of the maximally allowable mass M and radius R for isotropic fluid spheres of the form $2M/R \leq 8/9$ (where natural units $c = G = 1$ have been used).

The existence of the cosmological constant modifies the allowed ranges for various physical parameters, like, for example, the maximum mass of compact stellar objects, thus leading to a modification of the "classical" Buchdahl limit [18], for the effect of anisotropy, see e.g. [19].

The maximum allowable mass-radius ratio in the case of stable charged compact general relativistic objects was obtained in [20], by generalising to the charged case the methods used for neutral stars by Buchdahl [17] and Straumann [21].

On the other hand, we cannot exclude *a priori* the possibility that the cosmological constant, as a manifestation of vacuum energy, may play an important role not only at galactic or cosmological scales, but also at the level of elementary particles (the very successful phenomenological bag model of hadrons requires the existence of the vacuum energy inside and outside strongly interacting particles). With the use of the generalised Buchdahl identity [18], it can be rigorously proven that the existence of a non-negative Λ imposes a lower bound on the mass M and density ρ of general relativistic objects of radius R, which is given by [22]

$$2M \geq \frac{8\pi\Lambda}{6}R^3, \qquad \rho = \frac{3M}{4\pi R^3} \geq \frac{\Lambda}{2} =: \rho_{\rm min}. \tag{1}$$

Therefore, the existence of the cosmological constant implies the existence of an absolute minimum mass and density in the universe. No object present in relativity can have a density that is smaller than $\rho_{\rm min}$. For $\Lambda > 0$ this result also implies a minimum density for stable fluctuations in energy density.

It is the purpose of the present paper to consider the problem of the existence of a minimum mass-radius ratio for compact electrically charged general relativistic objects. We rigorously prove that a lower bound for the ratio M/R does exist for charged objects with non-zero electric charge Q. This result follows from the same Buchdahl type inequality which has been extensively used for the proof of the existence of an upper bound for the mass-radius ratio.

The present paper is organised as follows. The generalised Buchdahl inequality for charged objects in the presence of a vacuum energy (a cosmological constant) is derived in Section II. In Section III we obtain some bounds on the total charge and mass of compact charged objects from the study of the Ricci scalar invariants. The total energy (including the gravitational one) and the stability of the objects with minimum mass-radius ratio is considered in Section IV. We discuss and conclude our results in Section V.

Throughout this paper we use the Landau-Lifshitz conventions [23] for the metric signature $(+,-,-,-)$ and for the field equations, and a system of units with $c = G = \hbar = 1$.

II. GENERALISED BUCHDAHL INEQUALITY FOR CHARGED OBJECTS

For a static general relativistic spherically symmetric configuration the interior line element is given by

$$ds^2 = e^{\nu(r)}dt^2 - e^{\lambda(r)}dr^2 - r^2\left(d\theta^2 + \sin^2\theta\, d\varphi^2\right). \tag{2}$$

The properties of a charged compact general relativistic object can be completely described by the structure equations, which are given by

$$\frac{dm}{dr} = 4\pi\rho r^2 + \frac{Q}{r}\frac{dQ}{dr}, \tag{3}$$

$$\frac{dp}{dr} = -\frac{(\rho + p)\left[m + 4\pi r^3\left(p - \frac{2B}{3}\right) - \frac{Q^2}{r}\right]}{r^2\left(1 - \frac{2m}{r} + \frac{Q^2}{r^2} - \frac{8\pi}{3}Br^2\right)} + \frac{Q}{4\pi r^4}\frac{dQ}{dr}, \tag{4}$$

$$\frac{d\nu}{dr} = \frac{2\left[m + 4\pi r^3\left(p - \frac{2B}{3}\right) - \frac{Q^2}{r}\right]}{r^2\left(1 - \frac{2m}{r} + \frac{Q^2}{r^2} - \frac{8\pi}{3}Br^2\right)}, \tag{5}$$

where $\rho(r)$ is the energy density of the matter, $p(r)$ is the thermodynamic pressure, $m(r)$ is the mass and

$$Q(r) = 4\pi \int_0^r e^{\frac{\nu+\lambda}{2}} r'^2 j^0 dr', \tag{6}$$

is the electric charge inside radius r, respectively. The electric current inside the charged object is given by $j^\mu = (j^0, 0, 0, 0)$. By analogy with the bag model of hadrons we also assume the presence of an effective constant vacuum energy density B (a cosmological constant) inside and outside the charged object. Eqs. (3)–(5) represent the generalisation of the structure equations for general relativistic static charged objects, introduced for the first time in [24], by taking into account the existence of a non-zero vacuum energy.

Generally p and ρ are related by an equation of state of the form $\rho = \rho(p)$. The structure equations Eqs. (3)–(5) must be considered together with the boundary conditions $p(R) = 0$, $p(0) = p_c$, $\rho_c = \rho(p = 0)$ and $Q(0) = 0$, where ρ_c, p_c are the central density and pressure, respectively.

With the use of Eqs. (3)–(5) it is easy to show that the function $\zeta = \exp(\nu/2) > 0$, $\forall r \in [0, R]$, obeys the equation

$$\sqrt{1 - \frac{2m}{r} + \frac{Q^2}{r^2} - \frac{8\pi}{3}Br^2}\frac{1}{r}\frac{d}{dr}\left[\sqrt{1 - \frac{2m}{r} + \frac{Q^2}{r^2} - \frac{8\pi}{3}Br^2}\frac{1}{r}\frac{d\zeta}{dr}\right] = \frac{\zeta}{r}\left[\frac{d}{dr}\frac{m}{r^3} + \frac{Q^2}{r^5}\right]. \tag{7}$$

For $Q = 0$ and $B = 0$ we obtain the equation considered in [21]. Since the density ρ does not increase with increasing r, the mean density of the matter $\langle\rho\rangle = 3m(r)/4\pi r^3$ inside radius r does not increase either. Therefore we assume that inside a compact general relativistic object the condition

$$\frac{d}{dr}\frac{m}{r^3} < 0, \tag{8}$$

holds, independently of the equation of state of dense matter and of the electric charge distribution inside the object.

By defining a new function

$$\eta(r)=\int_0^r \frac{r'}{\sqrt{1-\frac{2m(r')}{r'}+\frac{Q^2(r')}{r'^2}-\frac{8\pi}{3}Br'^2}}\left[\int_0^{r'}\frac{Q^2\left(r''\right)\zeta\left(r''\right)}{r''^5\sqrt{1-\frac{2m(r'')}{r''}+\frac{Q^2(r'')}{r''^2}-\frac{8\pi}{3}Br''^2}}dr''\right]dr', \tag{9}$$

denoting $\Psi=\zeta-\eta$, and introducing a new independent variable $\xi\left(r\right)$ by means of the transformation [20, 21]

$$\xi\left(r\right)=\int_0^r r'\left[1-\frac{2m(r')}{r'}+\frac{Q^2\left(r'\right)}{r'^2}-\frac{8\pi}{3}Br'^2\right]^{-\frac{1}{2}}dr', \tag{10}$$

from Eq. (9) we obtain the basic result that inside all stable stellar type charged general relativistic matter distributions the condition

$$\frac{d^2\Psi}{d\xi^2}<0, \tag{11}$$

must hold for all $r\in[0,R]$. Using the mean value theorem [21] we conclude that

$$\frac{d\Psi}{d\xi}\leq\frac{\Psi\left(\xi\right)-\Psi(0)}{\xi}, \tag{12}$$

or, taking into account that $\Psi(0)>0$ it follows that,

$$\Psi^{-1}\frac{d\Psi}{d\xi}\leq\frac{1}{\xi}. \tag{13}$$

In the following we denote

$$\alpha(r)=1-\frac{Q^2(r)}{2m(r)r}+\frac{4\pi}{3}B\frac{r^3}{m(r)}. \tag{14}$$

In the initial variables the inequality (13) takes the form

$$\frac{1}{r}\sqrt{1-\frac{2\alpha\left(r\right)m(r)}{r}}\left\{\frac{1}{2}\frac{d\nu}{dr}e^{\frac{\nu(r)}{2}}-\frac{r}{\sqrt{1-\frac{2\alpha(r)m(r)}{r}}}\int_0^r\frac{Q^2\left(r'\right)e^{\frac{\nu(r')}{2}}}{r'^5\sqrt{1-\frac{2\alpha(r')m(r')}{r'}}}dr'\right\}\leq$$
$$\frac{e^{\frac{\nu(r)}{2}}-\int_0^r r'\left[1-\frac{2\alpha(r')m(r')}{r'}\right]^{-\frac{1}{2}}\left\{\int_0^{r'}\left[1-\frac{2\alpha(r'')m(r'')}{r''}\right]^{-\frac{1}{2}}\frac{Q^2(r'')e^{\frac{\nu(r'')}{2}}}{r''^5}dr''\right\}dr'}{\int_0^r r'\left[1-\frac{2\alpha(r')m(r')}{r'}\right]^{-\frac{1}{2}}dr'}. \tag{15}$$

For any stable charged compact objects m/r^3 does not increase outwards. We suppose that for all $r' \leq r$ we have

$$\frac{\alpha\left(r'\right)m(r')}{r'} \geq \frac{\alpha\left(r\right)m(r)}{r}\left(\frac{r'}{r}\right)^2, \tag{16}$$

or, equivalently,

$$\frac{2m\left(r'\right)}{r'} - \frac{2m(r)}{r}\left(\frac{r'}{r}\right)^2 \geq \frac{Q^2(r')}{r'^2} - \frac{Q^2(r)}{r^2}\left(\frac{r'}{r}\right)^2. \tag{17}$$

We also assume that inside the compact stellar object the charge $Q(r)$ satisfies the general condition

$$\frac{Q^2(r'')e^{\frac{\nu(r'')}{2}}}{r''^5} \geq \frac{Q^2(r')e^{\frac{\nu(r')}{2}}}{r'^5} \geq \frac{Q^2(r)e^{\frac{\nu(r)}{2}}}{r^5}, r'' \leq r' \leq r. \tag{18}$$

Therefore, we can evaluate the terms in Eq. (15) as follows. For the term in the denominator of the right hand side of Eq. (15) we obtain:

$$\left\{\int_0^r r'\left[1-\frac{2\alpha\left(r'\right)m\left(r'\right)}{r'}\right]^{-\frac{1}{2}}dr'\right\}^{-1} \leq \frac{2\alpha(r)m(r)}{r^3}\left[1-\sqrt{1-\frac{2\alpha(r)m(r)}{r}}\right]^{-1}. \tag{19}$$

For the second term in the bracket of the left hand side of Eq. (15) we have

$$\begin{aligned}
&\int_0^r \left[1-\frac{2\alpha\left(r'\right)m\left(r'\right)}{r'}\right]^{-\frac{1}{2}}\frac{Q^2\left(r'\right)e^{\frac{\nu(r')}{2}}}{r'^5}dr' \\
&\qquad \geq \frac{Q^2(r)e^{\frac{\nu(r)}{2}}}{r^5}\int_0^r\left[1-\frac{2\alpha\left(r\right)m\left(r\right)}{r}\left(\frac{r'}{r}\right)^2\right]^{-\frac{1}{2}}dr' \\
&\qquad = \frac{Q^2(r)e^{\frac{\nu(r)}{2}}}{r^5}\left[\frac{2\alpha(r)m(r)}{r^3}\right]^{-\frac{1}{2}}\arcsin\left[\sqrt{\frac{2\alpha(r)m(r)}{r}}\right].
\end{aligned} \tag{20}$$

The second term in the nominator of the right hand side of Eq. (15) can be evaluated as

$$\begin{aligned}
&\int_0^r r'\left[1-\frac{2\alpha\left(r'\right)m\left(r'\right)}{r'}\right]^{-\frac{1}{2}}\left\{\int_0^{r'}\left[1-\frac{2\alpha\left(r''\right)m\left(r''\right)}{r''}\right]^{-\frac{1}{2}}\frac{Q^2\left(r''\right)e^{\frac{\nu(r'')}{2}}}{r''^5}dr''\right\}dr' \\
&\geq \int_0^r r'\left[1-\frac{2\alpha\left(r'\right)m\left(r'\right)}{r'}\right]^{-\frac{1}{2}}\frac{Q^2\left(r'\right)e^{\frac{\nu(r')}{2}}}{r'^4}\left[\frac{2\alpha(r')m(r')}{r'}\right]^{-\frac{1}{2}}\arcsin\left[\sqrt{\frac{2\alpha(r')m(r')}{r'}}\right]dr' \\
&\geq \frac{Q^2(r)e^{\frac{\nu(r)}{2}}}{r^5}\int_0^r r'^2\left[1-\frac{2\alpha(r)m(r)}{r^3}r'^2\right]^{-\frac{1}{2}}\left[\frac{2\alpha(r)m(r)}{r^3}r'^2\right]^{-\frac{1}{2}}\arcsin\left[\sqrt{\frac{2\alpha(r)m(r)}{r^3}}r'\right]dr' \\
&\quad = \frac{Q^2(r)e^{\frac{\nu(r)}{2}}}{r^{\frac{1}{2}}(2\alpha(r)m(r))^{\frac{3}{2}}}\left\{\sqrt{\frac{2\alpha(r)m(r)}{r}}-\sqrt{1-\frac{2\alpha(r)m(r)}{r}}\arcsin\left[\sqrt{\frac{2\alpha(r)m(r)}{r}}\right]\right\}.
\end{aligned} \tag{21}$$

In order to obtain the inequality (21) we have also used the property of monotonic increase in the interval $x \in [0,1]$ of the function $\arcsin x/x$.

Using Eqs. (19)–(21), Eq. (15) becomes:

$$\left[1-\sqrt{1-\frac{2\alpha(r)m(r)}{r}}\right]\frac{m(r)+4\pi r^3\left(p-\frac{2}{3}B\right)-\frac{Q^2}{r}}{r^3\sqrt{1-\frac{2\alpha(r)m(r)}{r}}}$$
$$\leq \frac{2\alpha(r)m(r)}{r^3}+\frac{Q^2}{r^4}\left\{\frac{\arcsin\left[\sqrt{\frac{2\alpha(r)m(r)}{r}}\right]}{\sqrt{\frac{2\alpha(r)m(r)}{r}}}-1\right\}. \quad (22)$$

The Buchdahl type inequality given by Eq. (22) is valid for all r inside the electrically charged object. It naturally leads to the existence of a maximum mass-radius ratio for general relativistic objects.

Consider first the neutral case $Q = 0$ and assume that the vacuum energy is zero, $B = 0$. We assume that at the surface of the compact object, defined by a radius $r = R$, the thermodynamical pressure p vanishes, $p(R) = 0$. By evaluating (22) for $r = R$ we obtain $(1-2M/R)^{-1/2} \leq 2\left[1-(1-2M/R)^{-1/2}\right]^{-1}$, leading to the well-known result $2M/R \leq 8/9$ [17, 21]. The maximum mass-radius ratio for charged object, representing the generalisation to the charged case of the Buchdahl limit, was considered, and extensively discussed, in the case of a vanishing vacuum energy $B = 0$, in [20].

III. MINIMUM MASS-RADIUS RATIO FOR CHARGED GENERAL RELATIVISTIC OBJECTS

Eq. (22) also implies the existence of a minimum mass-radius ratio for compact charged general relativistic objects. This can be shown as follows. For small values of the argument the function $\arcsin x/x - 1$ can be approximated as $\arcsin x/x - 1 \approx x^2/6$. Therefore, at the vacuum boundary $r = R$ of the charged object, Eq. (22) can be written in an equivalent form as

$$\sqrt{1-\frac{2M}{R}+\frac{Q^2}{R^2}-\frac{8\pi}{3}BR^2} \geq \frac{M-\frac{Q^2}{R}-\frac{8\pi}{3}BR^3}{3M-2\frac{Q^2}{R}+\frac{Q^2}{6R^2}\left(2M-\frac{Q^2}{R}+\frac{8\pi}{3}BR^3\right)}. \quad (23)$$

By introducing a new variable u defined as

$$u=\frac{M}{R}-\frac{Q^2}{2R^2}+\frac{4\pi}{3}BR^2, \quad (24)$$

Eq. (23) takes the form

$$\sqrt{1-2u} \geq \frac{u-a}{bu-a}, \tag{25}$$

where we denoted

$$a = \frac{Q^2}{2R^2} + 4\pi B R^2, \tag{26}$$

and

$$b = 3 + \frac{Q^2}{3R^2}, \tag{27}$$

respectively. Then, by squaring, we can reformulate the condition given by Eq. (25) as

$$u\left[2b^2u^2 - \left(b^2 + 4ab - 1\right)u + 2a\left(a + b - 1\right)\right] \leq 0, \tag{28}$$

or, equivalently,

$$u\left(u - u_1\right)\left(u - u_2\right) \leq 0, \tag{29}$$

where

$$u_1 = \frac{b^2 + 4ab - 1 - (1-b)\sqrt{(1+b)^2 - 8ab}}{4b^2}, \tag{30}$$

and

$$u_2 = \frac{b^2 + 4ab - 1 + (1-b)\sqrt{(1+b)^2 - 8ab}}{4b^2}, \tag{31}$$

respectively.

Since $u \geq 0$, Eq. (29) is satisfied if $u \leq u_1$ and $u \geq u_2$, or $u \geq u_1$ and $u \leq u_2$. However, the condition $u \geq u_1$ contradicts the upper bound which follows from Eq. (22), and which has been discussed in detail in [20]. Therefore, Eq. (29) is satisfied if and only if for all values of the physical parameters the condition $u \geq u_2$ holds. This is equivalent to the existence of a minimum bound for the mass-radius ratio of compact anisotropic objects, which is given by

$$u \geq \frac{2a}{1+b}, \tag{32}$$

where we have taken into account that $(1+b)^2 \gg 8ab$. Using the expressions of a, b and u as defined above yields the minimum mass-radius ratio for electrically charged general relativistic objects as

$$\frac{2M}{R} \geq \frac{3}{2}\frac{Q^2}{R^2}\frac{1 + \frac{8\pi}{9}B\frac{R^4}{Q^2} - \frac{4\pi}{27}BR^2 + \frac{Q^2}{18R^2}}{1 + \frac{Q^2}{12R^2}}. \tag{33}$$

Let us neglect the dark energy component ($B = 0$) for the moment, then the minimum mass-radius ration (33) takes the following form

$$\frac{2M}{R} \geq \frac{3}{2}\frac{Q^2}{R^2}\frac{1+\frac{Q^2}{18R^2}}{1+\frac{Q^2}{12R^2}}, \tag{34}$$

which can be Taylor expanded in the term Q^2/R^2. The assumption $Q^2/R^2 \ll 1$ is natural since the total charge is always many orders smaller than the radii of charged stellar objects. Therefore we find

$$\frac{2M}{R} \geq \frac{3}{2}\frac{Q^2}{R^2}\Big(1 - \frac{Q^2}{36R^2} + O(Q^2/R^2)^4\Big), \tag{35}$$

that in the lowest order in Q^2/R^2 the mass-radius ration is bounded from below by $2M/R \geq 3Q^2/2R^2$. For $Q = 0$ and $B \neq 0$ the minimum mass for neutral objects in the presence of the vacuum energy is found, see (1) in the Introduction.

If in equation (33) we neglect the term containing the product BQ^2 and again assume that $Q^2/R^2 \ll 1$, the minimum mass of a charged particle can be generally represented in an approximate form as

$$M \geq \frac{4\pi}{6}BR^3 + \frac{3}{4}\frac{Q^2}{R}. \tag{36}$$

Furthermore, the mass of a spherically symmetric object can be written in terms of its mean density

$$\langle\rho\rangle \geq \frac{B}{2} + \frac{9}{16\pi}\frac{Q^2}{R^4}, \tag{37}$$

which represents a lower bound on the mean density. It should be noted that in the absence of charge, the lower bound (37) only depends on the dark energy component B and is independent of the object's radius R. Hence, the bound due to dark energy must be regarded as an absolute bound, valid on all scales of interest. On the other hand, the additional contribution on the minimal density due to the presence of charge depends on the radius. For large astrophysical objects, the additional charge term is suppressed by four orders of magnitude in the radius. Therefore, the charge term can only have an effect if relatively small objects and highly charge objects are considered. To further elucidate this point, let us introduce the surface charge density given by

$$\sigma = \frac{Q}{4\pi R^2}, \tag{38}$$

where it should be noted that the charge term Q takes the total charge of the stellar object into account. Using this definition, Eq. (37) leads to

$$\langle \rho \rangle \geq \frac{B}{2} + 9\sigma. \tag{39}$$

It is now obvious that the charge can have a significant effect on the allowed mean density of the stellar like object. In particular, configurations where the charge is mainly located near the surface of the object yield a strong lower bound on the mean density of those general relativistic objects.

IV. MASS-RADIUS RATIO CONSTRAINTS FROM THE RICCI INVARIANTS

In order to find a general restriction for the total charge Q a compact stable object can acquire in the presence of a cosmological constant we consider the behaviour of the three Ricci invariants

$$r_0 = g^{ij} R_{ij} = R, \qquad r_1 = R_{ij} R^{ij}, \qquad r_2 = R_{ijkl} R^{ijkl}, \tag{40}$$

respectively.

If the general static line element is regular, satisfying the conditions $e^{\nu(0)} = \text{constant} \neq 0$ and $e^{\lambda(0)} = 1$, then the Ricci invariants are also non-singular functions throughout the compact object. In particular for a regular space-time the invariants are non-vanishing at the origin $r = 0$. For the invariant r_2 we find

$$\begin{aligned} r_2 = \left[8\pi\left(\rho + p\right) - \frac{4m}{r^3} - \frac{16\pi}{3}B + \frac{6Q^2}{r^4}\right]^2 + 2\left(8\pi p + \frac{2m}{r^3} - \frac{16\pi}{3}B - \frac{2Q^2}{r^4}\right)^2 \\ + 2\left(8\pi\rho - \frac{2m}{r^3} + \frac{16\pi}{3}B + \frac{2Q^2}{r^4}\right)^2 + 4\left(\frac{2m}{r^3} + \frac{8\pi}{3}B - \frac{Q^2}{r^4}\right)^2. \end{aligned} \tag{41}$$

For a monotonically decreasing interior electric field $Q^2/8\pi r^4$, the function r_2 is regular and monotonically decreasing throughout the star. Therefore it satisfies the condition $r_2(R) < r_2(0)$, leading to the following equation quadratic in Q^2/R^4

$$\begin{aligned} \left(\frac{Q^2}{R^4}\right)^2 + \left(\frac{Q^2}{R^4}\right)\frac{16\pi}{7}B - \frac{24}{7}\pi^2 p_c^2 - \frac{16}{7}\pi^2 p_c \rho_c - \frac{40}{21}\pi^2 \rho_c^2 \\ + \frac{32}{21}\pi^2 \langle\rho\rangle^2 + \frac{32}{7}\pi^2 p_c B - \frac{32}{21}\pi^2 \rho_c B < 0, \end{aligned} \tag{42}$$

where we assumed that at the surface of the star the matter density vanishes, $\rho(R) = 0$. We rewrite this in the form

$$\left(\frac{Q^2}{R^4} - q_+\right)\left(\frac{Q^2}{R^4} - q_-\right) < 0, \tag{43}$$

where the two roots are given by

$$q_\pm = -\frac{24\pi B}{21} \pm \frac{2\pi\rho_c\sqrt{6}}{21}\sqrt{35+42\frac{p_c}{\rho_c}\left(1-\frac{2B}{\rho_c}\right)+63\frac{p_c^2}{\rho_c^2}-28\frac{\langle\rho\rangle^2}{\rho_c^2}+28\frac{B}{\rho_c}+24\frac{B^2}{\rho_c^2}}. \quad (44)$$

Since the term Q^2/R^4 is positive definite, Eq. (43) can only be satisfied if

$$q_- < \frac{Q^2}{R^4} \quad \text{and} \quad q_+ > \frac{Q^2}{R^4}. \quad (45)$$

This first condition is simply the positivity of Q^2/R^4, whereas the second condition yields the upper bound

$$\frac{Q^2}{R^4} < \frac{2\pi\rho_c\sqrt{6}}{21}\sqrt{35+42\frac{p_c}{\rho_c}\left(1-\frac{2B}{\rho_c}\right)+63\frac{p_c^2}{\rho_c^2}-28\frac{\langle\rho\rangle^2}{\rho_c^2}+28\frac{B}{\rho_c}+24\frac{B^2}{\rho_c^2}}-\frac{24\pi B}{21}, \quad (46)$$

which for vanishing dark energy simplifies to

$$\frac{Q^2}{R^4} < \frac{2\pi\rho_c\sqrt{6}}{21}\sqrt{35+42\frac{p_c}{\rho_c}+63\frac{p_c^2}{\rho_c^2}-28\frac{\langle\rho\rangle^2}{\rho_c^2}}. \quad (47)$$

Another condition on $Q(R)$ can be obtained from the study of the scalar

$$r_1 = \left(8\pi\rho + 8\pi B + \frac{Q^2}{r^4}\right)^2 + 3\left(8\pi p - 8\pi B - \frac{Q^2}{r^4}\right)^2 + \frac{64\pi p Q^2}{r^4} - \frac{64\pi B Q^2}{r^4}. \quad (48)$$

Under the same assumptions of regularity and monotonicity for the function r_1 and considering that the surface density is vanishing we obtain for the surface value of the monotonically decreasing electric field the upper bound

$$\frac{Q^2}{R^4} < 4\pi\rho_c\sqrt{1+3\frac{p_c^2}{\rho_c^2}+2\left(1-3\frac{p_c}{\rho_c}\right)\frac{B}{\rho_c}}. \quad (49)$$

For negligible dark energy ($B=0$) this condition becomes

$$\frac{Q^2}{R^4} < 4\pi\rho_c\sqrt{1+3\frac{p_c^2}{\rho_c^2}}. \quad (50)$$

Let us furthermore assume that the equation of state near the centre is stiff matter ($p=\rho$) or radiation ($p=\rho/3$) like, then for the respective cases Eq. (50) yields the two conditions

$$\sigma^2 < \frac{\rho_c}{2\pi}, \qquad \text{stiff matter}, \quad (51)$$

$$\sigma^2 < \frac{\rho_c}{2\pi\sqrt{3}}, \qquad \text{radiation}. \quad (52)$$

The invariant r_0 leads to the trace condition $\rho_c + B > 3p_c - 3B$ of the energy-momentum tensor that holds at the centre of the fluid spheres.

V. TOTAL ENERGY AND STABILITY OF CHARGED OBJECTS WITH MINIMUM MASS-RADIUS RATIO

As another application of the obtained minimum mass-radius ratio for charged objects we derive an explicit expression for the total energy of compact charged general relativistic objects with minimum mass-radius ratio.

The total energy E (including the gravitational field contribution) inside an equipotential surface S of radius R can be defined, according to [25], to be

$$E = E_M + E_F = \frac{1}{8\pi}\xi_s \int_S [K] dS, \tag{53}$$

where ξ^i is a Killing vector field of time translation, ξ_s its value at S and $[K]$ is the jump across the shell of the trace of the extrinsic curvature of S, considered as embedded in the 2-space $t =$ constant. $E_M = \int_S T_i^k \xi^i \sqrt{-g} dS_k$ and E_F are the energy of the matter and of the gravitational field, respectively, with T_i^k the energy-momentum tensor of the matter. This definition is manifestly coordinate invariant.

For a static charged spherically symmetric system in the presence of a cosmological constant the total energy inside the radius R is

$$E = R\left[1 - \left(1 - \frac{2M}{R} + \frac{Q^2}{R^2} - \frac{8\pi}{3}BR^2\right)^{1/2}\right]\left(1 - \frac{2M}{R} + \frac{Q^2}{R^2} - \frac{8\pi}{3}BR^2\right)^{1/2}. \tag{54}$$

For the minimum mass-radius ratio charged object, with $2M/R = (3/2)\,Q^2/R^2 + 4\pi BR^2/3$, the total energy can be expressed in terms of the radius, charge and vacuum energy only as

$$E = R\left[1 - \left(1 - \frac{Q^2}{2R^2} - 4\pi BR^2\right)^{1/2}\right]\left(1 - \frac{Q^2}{2R^2} - 4\pi BR^2\right)^{1/2}. \tag{55}$$

For a stable configuration, the energy should have a minimum,

$$\frac{\partial E}{\partial R} = 0, \tag{56}$$

a condition which gives the following algebraic equation determining R as a function of B and Q:

$$1 + \frac{Q^2}{2R^2} - 12\pi BR^2 + \frac{1 - 8\pi BR^2}{\sqrt{1 - \frac{Q^2}{2R^2} - 4\pi BR^2}} = 0. \tag{57}$$

By Taylor-expanding the square root and keeping only the first order terms in Q^2 and B we obtain the radius of the stable minimum mass charged configuration as

$$R = (24\pi)^{-1/4} \frac{\sqrt{Q}}{B^{1/4}}. \tag{58}$$

Therefore the minimum mass of a charged object can be expressed as a function of the vacuum energy density B and the electric charge in the form

$$M = \frac{7}{9}(24\pi)^{1/4} Q^{3/2} B^{1/4}. \tag{59}$$

By eliminating the vacuum energy between Eqs. (58) and (59) we obtain the following mass-radius-charge relation:

$$M = \frac{7}{9}\frac{Q^2}{R}. \tag{60}$$

The surface charge density of the stable objects with minimum mass-radius ratio can be expressed in terms of the vacuum energy only as

$$\sigma = \sqrt{\frac{3B}{2\pi}}. \tag{61}$$

VI. DISCUSSIONS AND FINAL REMARKS

In the present paper we have shown that a minimum mass-radius ratio for charged stable compact general relativistic objects do exist, and it is the direct consequence of the same Buchdahl inequality giving the upper bound for the mass-radius ratio. In the case of the minimum mass-radius ratio it is also possible to obtain explicit inequalities giving the lower bound for $2M/R$ as an explicit function of the charge Q and of the vacuum energy density B. The condition of the thermodynamic stability of the minimum mass object leads to an explicit representation of the mass and radius in terms of the charge Q and of the vacuum energy B only.

The results obtained in the present paper are general and they can be easily extended to the case of other dark energy models, like, for example, the phantom fluid case with negative energy density. In the simplest case we can model phenomenologically the phantom fluid as having an energy density $B < 0$. Then, as one can see from Eq. (36), a negative B will lead to a decrease in the mass of charged phantom-like particle. Since it is reasonable to assume the condition $M \geq 0$, we obtain a general constraint on the magnitude of the phantom energy density of the form $B \leq (9/8\pi)\, Q^2/R^4$. On the other hand, if the fluid is phantom like, then the mass should tend to zero in the big rip singularity [16]. Our results show that in general the phantom energy and also the charge contribute to the minimal energy density. Therefore, for arbitrary charged phantom fluid particles the mass cannot become zero. Actually some minimal energy objects should remain, even if their spatial extension is of the order of the Planck length. Hence, our work suggests that in the big rip singularity (which appears in scalar phantom or effective phantom theories) [16], some remnants will remain, asking in the end whether such a big rip can occur and is not stopped by quantum effects.

A very interesting and long debated question is the possible applicability of general relativity to describe elementary particles, and, in particular, the electron. In 1919 Einstein [26] suggested a modification of the geometrical terms of the gravitational field equations of general relativity with only the energy-momentum tensor of the electromagnetic field being present in place of the energy-momentum tensor of matter. In this theory the self-stabilising stresses are of non-electromagnetic origin, the gravitational forces providing the necessary stability of the electron and also contributing to its mass. However, the breaking of the vacuum energy inside and outside charged particles may provide an alternative mechanism for the stabilisation of the charged elementary particles.

With respect to the scaling of the parameters B and Q of the form $B \to kB$ and $Q \to lQ$, the minimum mass and radius have the following scaling behaviours:

$$R \to l^{3/2} k^{-1/4} R, \qquad M \to l^{3/2} k^{1/4} M. \tag{62}$$

For a constant charge $l = 1$ particles with different masses can be constructed for different values of the vacuum energy by starting from a minimum mass configuration.

In the case of an electron, with mass $m_e = 0.51$ MeV and charge $e = \alpha^{1/2} = 137^{-1/2}$, where α is the fine structure constant, from Eq. (59) it follows that the value of the vacuum energy B_e necessary to stabilise the configuration is $B_e^{1/4} = 8.91$ MeV. In the case of quarks and hadrons, the value of the vacuum energy (bag constant) necessary to stabilise the bag is $B_{QCD}^{1/4} = 145$ MeV [3]. On the other hand, the radius of the electron obtained with the use of $B_e^{1/4} = 8.91$ MeV, given by Eq. (58), is $R_e = 0.011$ MeV^{-1} $= 2.19$ fm (1 MeV $= 5.064 \times 10^{-3}$ fm^{-1}). Therefore Eqs. (58) and (59) can give a satisfactory description of the basic classical physical parameters of the electron.

By interpreting the charge Q in Eq. (59) as a generalised charge, we can apply it even for strongly interacting particles. In the case of strong interactions, the strong coupling constant α_s is a function of the particle momenta. The quark-quark-gluon coupling constant for the simplest hadrons is $\alpha_s \approx 0.12$, and, by defining the generalised charge as $Q_{QCD} \approx \alpha_s^{1/2}$, with the use of the value of the bag constant as obtained in quantum chromodynamics, we obtain for the mass of the quarks a reasonable value of the order of $m_q = 67.75$ MeV.

The possibility that general relativity or a similar geometric description may play an important role at the scale of elementary particles is still very controversial. On the other hand, the possibility of the estimation of the mass of the charged elementary particles from general relativistic considerations in the framework of a broken vacuum model can perhaps give a better understanding of the deep connection between micro- and macro-physics.

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